ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Math. Anal. Appl. 294 (2004) 548–559

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Limit-point type solutions of nonlinear differential equations

Octavian G. Mustafa [a] and Yuri V. Rogovchenko [b,*]

[a] Centre for Nonlinear Analysis, University of Craiova, Al. I. Cuza 13, Craiova, Romania

[b] Department of Mathematics, Eastern Mediterranean University, Famagusta, TRNC, Mersin 10, Turkey

Received 10 December 2002

Available online 2 April 2004

Submitted by H.L. Smith

Abstract

We are concerned with the nonexistence of L^2-solutions of a nonlinear differential equation $x'' = a(t)x + f(t,x)$. By applying technique similar to that exploited by Hallam [SIAM J. Appl. Math. 19 (1970) 430–439] for the study of asymptotic behavior of solutions of this equation, we establish nonexistence of solutions from the class $L^2(t_0,\infty)$ under milder conditions on the function $a(t)$ which, as the examples show, can be even square integrable. Therefore, the equation under consideration can be classified as of limit-point type at infinity in the sense of the definition introduced by Graef and Spikes [Nonlinear Anal. 7 (1983) 851–871]. We compare our results to those reported in the literature and show how they can be extended to third order nonlinear differential equations.
© 2004 Elsevier Inc. All rights reserved.

Keywords: Nonlinear differential equations; Second order; Limit-point/limit-circle classification; Square integrable solutions

1. Introduction

The problem of finding sufficient conditions which guarantee that nth order linear and nonlinear ordinary differential equations do not possess nontrivial L^2-solutions is of special interest and has attracted the attention of numerous researchers. The systematic study of this problem has been initiated by Wintner in the pioneering paper [25] concerned with the linear differential equation

* Corresponding author.
E-mail address: yuri.rogovchenko@emu.edu.tr (Yu.V. Rogovchenko).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.02.029

$$x'' + a(t)x = 0, \quad t \geqslant t_0 \geqslant 0, \tag{1}$$

where $a(t)$ is a continuous real-valued function. Wintner's results for Eq. (1) have been refined later by Grammatikopoulos and Kulenović [13] and have been extended to the nonlinear equation

$$x'' + a(t)x^p = 0, \quad p \geqslant 1,$$

by Suyemoto and Waltman [23], and to the nonlinear equation

$$x'' + a(t)f(x) = 0, \quad t \geqslant t_0 \geqslant 0, \tag{2}$$

where $a(t)$ and $f(x)$ are continuous real-valued functions, by Burlak [6], Detki [8], and Grammatikopoulos and Kulenović [13]. A number of interesting theorems on the nonexistence of L^2-solutions for other classes of linear and nonlinear equations can be also found in papers by Eliaš [9], Grammatikopoulos and Kulenović [13], Hallam [14], Wong [27] and Zettl [28]. Some new results in this direction have been obtained by the authors [20,21]. We also mention the recent paper by the authors [19], where nonexistence of nontrivial square integrable solutions for the nth order nonlinear differential equation

$$x^{(n)} + f(t,x) = 0$$

has been established by using an estimate for the rate of decay of the L^2-norm of solutions of the perturbed equation

$$x^{(n)} + f(t,x) = b(t).$$

On the other hand, the problem of nonexistence of square integrable solutions is closely related to the limit-circle/limit-point classification originating from the work by Weyl [24] on spectral theory of Sturm–Liouville problems on infinite intervals. Given a second order linear differential equation

$$-\big(p(t)x'\big)' + q(t)x = \lambda x, \quad \lambda \in \mathbb{C}, \tag{3}$$

on an interval $I = [t_0, \infty)$, we say that the operator $\mathcal{L}$ defined by

$$\mathcal{L}x = -\big(p(t)x'\big)' + q(t)x$$

is of the *limit-circle type* at infinity if for a particular complex number λ_0 every solution $\varphi(t)$ of Eq. (3) satisfies

$$\int_{t_0}^{\infty} \big|\varphi(t)\big|^2\,dt < +\infty,$$

otherwise, $\mathcal{L}$ is said to be of the *limit-point type* at infinity (cf. Coddington and Levinson [7, Chapter 9] and Kauffman et al. [17, Chapter 3]). This definition has been extended to higher order linear differential equations, second order nonlinear differential equations, and higher order nonlinear differential equations (see, for instance, Atkinson [1], Bartušek and Graef [2], Bartušek et al. [3–5], Graef [10], Graef and Spikes [11,12], and references cited therein).

In this paper, we use the definition given by Graef and Spikes [11] classifying a perturbed second order nonlinear equation

$$\big(r(t)x'\big)' + q(t)f(x) = r(t,x)$$

to be of the *limit-circle type* if, for any solution $\varphi(t)$, either

$$\int_{t_0}^{\infty} \varphi(t) f\big(\varphi(t)\big)\,dt < +\infty$$

or

$$\int_{t_0}^{\infty} F\big(\varphi(t)\big)\,dt < +\infty,$$

where

$$F(u) = \int_{t_0}^{u} f(u)\,du,$$

and to be of the *limit-point type* otherwise. We note that numerous papers not only dealt with the problem of limit-point/limit-circle classification of linear and nonlinear differential equations, but discussed such important issues as continuation of solutions and asymptotic behavior of solutions, including boundedness, oscillation, convergence to zero, asymptotic expansions for solutions, etc. We refer the reader to the recent papers by Bartušek et al. [3–5], Graef [10], Graef and Spikes [11,12], where further references can be found.

The purpose of this paper is to study the nonexistence of L^2-solutions of the nonlinear differential equation

$$x'' = a(t)x + f(t,x), \quad t \geqslant t_0 \geqslant 1, \tag{4}$$

on the interval $(t_0, +\infty)$ under assumptions that $f(t,x)$ satisfies a Lipschitz-type condition with respect to the second argument and $a(t)$ satisfies some technical conditions specified in the sequel. Our main result establishes sufficient conditions which guarantee that the only square integrable solution of Eq. (4) is the trivial one, and this enables one to classify Eq. (4) as of limit-point type at infinity.

We stress that, as opposed to the results obtained by Grammatikopoulos and Kulenović [13] under the hypothesis that

$$\int^{\infty} t^2 \big|a(t)\big|^2\,dt < +\infty, \tag{5}$$

as well as to those derived by Burlak [6], Detki [8], Eliaš [9], Hallam [14], Suyemoto and Waltman [23], and Wong [26] by requesting even a stronger condition

$$\int^{\infty} t^3 \big|a(t)\big|^2\,dt < +\infty, \tag{6}$$

the assumptions we impose in this paper on the function $a(t)$ are satisfied for certain functions that belong to the class $L^2(0,\infty)$ (see the examples in the last section and also cf. [13, p. 135]). For an interesting discussion revealing the significance of the integrability assumption on coefficient $a(t)$, the reader is invited to consult the paper by Patula and Waltman [22] and references therein. We note that condition (6) has also been used by Hartman and Wintner [16] for the study of the existence of nonoscillatory solutions of linear differential equations.

For the convenience of the reader, in what follows we briefly adapt the results obtained by Grammatikopoulos and Kulenović [13] and Wong [26] for the second order differential equations (1) and (2).

Theorem 1 (cf. [13, Theorem, p. 132]). *Let the function $a(t)$ satisfy conditions* (5) *and*

$$\int^{\infty}\left(\int_t^{\infty}\left(\int_s^{\infty}\left|a(\tau)\right|^2 d\tau\right)^{1/2} ds\right)^2 dt < +\infty. \tag{7}$$

Then Eq. (1) *cannot have a nontrivial L^2-solution.*

The second hypothesis of Theorem 1 is rather technical, although Grammatikopoulos and Kulenović [13, p. 135] have shown that (7) is weaker than the assumption (6).

The following result, adapted from the paper by Wong [26], is concerned with the nonlinear equation (2).

Theorem 2 (cf. [26, Theorem, p. 104]). *Let the function $a(t)$ satisfy* (6) *and suppose that the function $f(u)$ is continuous on $\mathbb{R}$ and such that*

(i) $f(u) \neq 0$ *whenever* $u \neq 0$;
(ii) $\liminf_{|u|\to+\infty} f(u) \geqslant \alpha > 0$;
(iii) *for every L^2-solution $x(t)$ of Eq.* (2),

$$\limsup_{t\to+\infty}\left(\int_t^{\infty}\left|f\big(x(s)\big)\right|^2 ds\right)^{-1}\left(\int_t^{\infty}\left|x(s)\right|^2 ds\right) = \gamma > 0.$$

Then Eq. (2) *cannot have a nontrivial solution $x(t)$ for which*

$$\int^{\infty}\left|f\big(x(t)\big)\right|^2 dt < +\infty. \tag{8}$$

We point out that one of essential features of the proofs of Theorems 1 and 2 is that any nontrivial solution $x(t)$ of Eqs. (1) and (2) which satisfies (8) should also satisfy the following two conditions:

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x'(t) = 0 \tag{9}$$

(this is essentially [26, Lemma, p. 104], cf. also [25, p. 349]), and

$$\int_{T_0}^{\infty} |x(s)|^2\,ds \leqslant \left(\int_{T_0}^{\infty} \left|f\big(x(s)\big)\right|^2 ds\right)\left(\int_{T_0}^{\infty}\left(\int_{t}^{\infty}\left(\int_{s}^{\infty}|a(\tau)|^2\,d\tau\right)^{1/2} ds\right)^2 dt\right) \tag{10}$$

for every $T_0 \geqslant t_0$ (cf. [13, p. 134], where the inequality for a solution of an nth order equation is presented).

Clearly, condition (10) yields that any nontrivial solution $x(t)$ of Eq. (2) satisfying (8) belongs to the class $L^2(T_0,\infty)$. Thus, if this equation does not have nontrivial L^2-solutions, it has no solutions satisfying (8). On the other hand, condition (iii) in Theorem 2 implies that there exists a strictly increasing sequence $(t_n)_{n\geqslant 1}$, diverging to $+\infty$, such that

$$\int_{t_n}^{\infty} \left|f\big(x(s)\big)\right|^2 ds < \frac{2}{\gamma}\int_{t_n}^{\infty} |x(s)|^2\,ds, \quad n \geqslant 1, \tag{11}$$

and if $x(t)$ were square integrable, according to (11), $f(x(t))$ would enjoy the same property. Therefore, if $a(t)$ satisfies conditions of Theorem 1, solution $x(t)$ of Eq. (2) is square integrable if and only if $f(x(t))$ is square integrable, and it suffices to study existence of L^2-solutions of Eq. (2). We refer to the papers by Burlak [6, p. 262], Hallam [14, p. 133], Suyemoto and Waltman [23, p. 970], where similar reasoning can be found.

Detki [8, p. 115] assumed for Eq. (2) that $f \in C^1(\mathbb{R},\mathbb{R})$, $f(0)=0$, $f'(0)>0$, and $f'(x)\geqslant 0$ for all real x. Then it follows that

$$\alpha|x| \leqslant \left|f(x)\right| \leqslant \beta|x| \quad \text{for all } x \text{ small enough,}$$

where $\alpha,\beta>0$. If $x(t)$ is a solution of Eq. (2) satisfying $\lim_{t\to+\infty} x(t)=0$,

$$\alpha\int_{T}^{\infty} |x(t)|^2\,dt \leqslant \int_{T}^{\infty}\left|f\big(x(t)\big)\right|^2 dt \leqslant \beta\int_{T}^{\infty}|x(t)|^2\,dt \leqslant +\infty$$

for all $T>t_0$ large enough, which yields a similar conclusion for solutions that satisfy (9); see also the discussion in [6, p. 258].

Using similar reasoning, Grammatikopoulos and Kulenović [13] established the nonexistence of L^2-solutions of Eq. (2) combining the assumptions of Theorem 2 with the hypothesis (7) of Theorem 1.

We conclude this brief survey of the results on nonexistence of L^2-solutions by observing that the conclusions of Theorems 1 and 2 remain intact if one assumes that the functions $a(t)$ and $x(t)$ are complex-valued rather than real-valued functions.

2. Main result

Consider now the nonlinear differential equation (4). In what follows, we assume that

(A1) $x(t)$, $a(t)$ and $f(t,x)$ are complex-valued functions;
(A2) f is continuous on $J\times\mathbb{C}$, $J=[t_0,+\infty)$;
(A3) f satisfies a Lipschitz-type condition

$$\left|f(t,y_1)-f(t,y_2)\right| \leqslant w(t)|y_1-y_2|$$

for all $t \in J$ and for any $y_1, y_2 \in \mathbb{C}$, where $w(t)$ is a continuous nonnegative function defined on J;

(A4) $f(t,0) \equiv 0$ for t in J;

(A5) $a(t)$ is a continuous function on J.

To simplify the presentation of our results, we define the functions b and c by

$$b(t) \stackrel{\text{def}}{=} \int_t^\infty a(s)\,ds, \qquad c(t) \stackrel{\text{def}}{=} \int_t^\infty b(s)\,ds.$$

We mention that condition (9), essential for the results obtained by Grammatikopoulos and Kulenović [13], allows us to study the question of nonexistence of L^2-solutions of Eq. (4) by applying a technique similar to that used by Hallam in [15] for solving a problem of a different nature.

Theorem 3. *Assume that*

$$\int^\infty t\big|b(t)\big|^2\,dt < +\infty, \qquad \int^\infty \big|b(t)\big|\,dt < +\infty,$$

and

$$\int^\infty t^3\big|c(t)a(t)\big|^2\,dt < +\infty, \qquad \int^\infty t^3\big[w(t)\big]^2\,dt < +\infty.$$

Then the only square integrable solution of Eq. (4) *satisfying condition* (9) *is the trivial one.*

Proof. Let $T_0 \geqslant t_0$. In what follows, we denote by X the Banach space $L^2((T_0,+\infty);\mathbb{C})$ of square-integrable complex-valued functions on (T_0,∞) endowed with the usual norm

$$\|x\|_{L^2} = \left(\int_{T_0}^\infty \big|x(s)\big|^2\,ds\right)^{1/2}.$$

Define now the integral operator $\mathcal{P}$ (cf. [15]) by the formula

$$(\mathcal{P}x)(t) = -c(t)x(t) + 2\int_t^\infty b(s)x(s)\,ds + \int_t^\infty (s-t)c(s)a(s)x(s)\,ds + \int_t^\infty (s-t)\big(1+c(s)\big)f\big(s,x(s)\big)\,ds,$$

where $t \geqslant T_0$ and $x \in X$.

We shall show that the operator $\mathcal{P}: X \to X$ is well defined. Applying the Cauchy–Schwarz inequality, we have for all $t \geqslant T_0$,

$$\int_t^\infty \left|b(s)x(s)\right| ds \leqslant \|x\|_{L^2} \left(\int_t^\infty \left|b(s)\right|^2 ds \right)^{1/2}, \tag{12}$$

$$\int_t^\infty (s-t)\left|c(s)a(s)x(s)\right| ds \leqslant \|x\|_{L^2} \left(\int_t^\infty \left(s\left|c(s)a(s)\right|\right)^2 ds \right)^{1/2}, \tag{13}$$

and

$$\int_t^\infty (s-t)\left(1+\left|c(s)\right|\right)\left|f\left(s,x(s)\right)\right| ds \leqslant \|x\|_{L^2}(1+M) \left(\int_t^\infty s^2\left[w(s)\right]^2 ds \right)^{1/2}, \tag{14}$$

where M is defined by

$$\int_{T_0}^\infty \left|b(s)\right| ds \leqslant M < +\infty.$$

Using inequalities (12)–(14), we obtain for the operator $\mathcal{P}$ the following estimate:

$$\begin{aligned}\int_{T_0}^\infty \left|(\mathcal{P}x)(t)\right|^2 dt \leqslant{}& 4\int_{T_0}^\infty \left|c(t)x(t)\right|^2 dt + 16\|x\|_{L^2}^2 \int_{T_0}^\infty \left(\int_t^\infty \left|b(s)\right|^2 ds \right) dt \\ &+ 4\|x\|_{L^2}^2 \int_{T_0}^\infty \left(\int_t^\infty s^2\left|c(s)a(s)\right|^2 ds \right) dt \\ &+ 4\|x\|_{L^2}^2 (1+M)^2 \int_{T_0}^\infty \left(\int_t^\infty s^2\left[w(s)\right]^2 ds \right) dt.\end{aligned}$$

We have to evaluate now the four integrals on the right-hand side of the latter inequality. First of all, since

$$\left|c(t)\right| \leqslant \int_{T_0}^\infty \left|b(s)\right| ds \leqslant M < +\infty, \quad t \geqslant T_0,$$

we have

$$\int_{T_0}^\infty \left|c(t)x(t)\right|^2 dt \leqslant M^2 \|x\|_{L^2}^2.$$

Passing to the second integral, we note that since

$$0 \leqslant \limsup_{t\to+\infty} t \int_t^\infty \left|b(s)\right|^2 ds \leqslant \lim_{t\to+\infty} \int_t^\infty s\left|b(s)\right|^2 ds = 0,$$

an integration by parts yields

$$\int_{T_0}^{\infty}\left(\int_{t}^{\infty}\bigl|b(s)\bigr|^2\,ds\right)dt=\int_{T_0}^{\infty}(t-T_0)\bigl|b(t)\bigr|^2\,dt<+\infty.$$

The estimates for the remaining two integrals are obtained analogously, and we conclude that the operator $\mathcal{P}$ is well defined on X.

Similar calculations for the norm of the operator $\mathcal{P}$ yield

$$\|\mathcal{P}x-\mathcal{P}y\|_{L^2}\leqslant L(T_0)\|x-y\|_{L^2}\quad \text{for all } x,y\in X, \tag{15}$$

where the constant $L(T_0)$ is given by the formula

$$\bigl[L(T_0)\bigr]^2=4\left(\int_{T_0}^{\infty}\bigl|b(t)\bigr|\,dt\right)^2+16\int_{T_0}^{\infty}t\bigl|b(t)\bigr|^2\,dt \\ +4\int_{T_0}^{\infty}t^3\bigl|c(t)a(t)\bigr|^2\,dt+4(1+M)^2\int_{T_0}^{\infty}t^3w^2(t)\,dt. \tag{16}$$

The fact that the nonlinearity $f(t,x)$ in Eq. (4) satisfies the Lipschitz-type condition (A3) with respect to the second argument is sufficient to guarantee global existence in the future of solutions of Eq. (4) (cf. Hallam [15] and Mustafa and Rogovchenko [18]; see also the references in the latter paper). Since solutions of Eq. (4) exist nonlocally, we can choose $T_0>t_0$ large enough to ensure that the constant $L(T_0)$ in (16) satisfies

$$0<L(T_0)<1. \tag{17}$$

It follows from (15) and (17) that the operator $\mathcal{P}$ is a contraction. Therefore, the operator equation $x=\mathcal{P}x$ has a unique solution in X. Clearly, this solution is trivial: $x=0$.

Suppose now, for the sake of obtaining a contradiction, that Eq. (4) has a nontrivial L^2-solution $x(t)$ on $(t_0,+\infty)$ which satisfies condition (9). We note that with the same choice of T_0 which guarantees that (17) holds, we have also $|c(t)|<1/2$ for all $t\geqslant T_0$. This means that $x(t)$ satisfies also the differential equation

$$\bigl(1+c(t)\bigr)x''=\bigl(1+c(t)\bigr)\bigl(a(t)x+f(t,x)\bigr). \tag{18}$$

Using condition (9), by repeated integration of Eq. (18) we conclude that $x(t)$ satisfies the integral equation

$$x(t)=(\mathcal{P}x)(t),\quad t\geqslant T_0,$$

but this equation has only the trivial solution, which is a contradiction. The proof is complete. □

Remark 1. Since the hypotheses of Theorem 3 are quite technical, eventual extensions of this result to higher order equations are rather intricate. Therefore, we proved the result on the nonexistence of L^2-solutions only for Eq. (4). Nevertheless, in the next section we shall explain how a similar result can be formulated and proved for a class of the third order nonlinear differential equations.

3. Discussion and further results

We start this section with the example showing that our assumptions on $a(t)$ are milder compared to those imposed by Grammatikopoulos and Kulenović [13]. Consider the function

$$a(t) = t^{-1}\exp(it^p),$$

where $p > 5$. Then we have for $t \geqslant t_0$ that

$$t^2\left|a(t)\right|^2 = 1, \qquad \left|b(t)\right| \leqslant 4p^{-1}t^{-p}.$$

It is easy to see that $a(t)$ satisfies all the hypotheses of Theorem 3 but fails to meet those of Theorem 1. Furthermore, it is not difficult to observe that $a \in L^2((T_0, +\infty); \mathbb{C})$.

Let us return to condition (9). Its presence in the conclusion of Theorem 3 makes our result less attractive than Theorem 1. Therefore, we would like to add a new hypothesis to the assumptions of Theorem 3 in order to ensure that any L^2-solution of Eq. (4) satisfies (9).

Corollary 4. *Assume that, in addition to the hypotheses of Theorem* 3*,* $a(t)$ *satisfies also the condition* (5)*. Then the only square integrable solution of Eq.* (4) *is the trivial one.*

Proof. Let $x(t)$ be an L^2-solution of Eq. (4). Proceeding as in the proof of the main result in [13] and making use of the assumption (5), we conclude that $x(t)$ satisfies condition (9). Then an application of Theorem 3 yields the desired conclusion. □

This proposition is useful for the classification of Eq. (4) as of the limit-point/limit-circle type discussed in the Introduction. Indeed, we have proved a much stronger assertion, and as an immediate consequence of Corollary 4, we obtain the following result.

Theorem 5. *Under the assumptions of Corollary* 4*, Eq.* (4) *is of the limit-point type at infinity in the sense of the definition of Graef and Spikes* [11]*.*

Consider now the function

$$a(t) = t^{-2}\exp(it^p),$$

where $p > 5$. A straightforward computation shows that $a(t)$ satisfies the conditions of Corollary 4 but fails to meet those of Theorem 1.

As it has been mentioned in the Introduction, hypothesis (7) of Theorem 1 is less restrictive than condition (6). Now, in order to examine the sharpness of the conditions in Corollary 4, let us consider, for a fixed $\varepsilon > 0$, the function a_ε defined by

$$a_\varepsilon(t) = t^{-(3+\varepsilon)/2}\exp(it^p),$$

where $p > 5$. It is easy to check that $a_\varepsilon(t)$ satisfies the hypotheses of Corollary 4 but does not satisfy those of Theorem 1. Furthermore,

$$\int^{\infty} t^{2+\varepsilon}\left|a_\varepsilon(t)\right|^2 dt = +\infty. \tag{19}$$

This example gives rise to the following question: since there are functions $a(t)$ which satisfy both (5) and (19), could condition (5) alone ensure the nonexistence of nontrivial L^2-solutions of Eq. (4)? The answer to this question is, unfortunately, negative. A relevant counter-example is provided by a simple linear differential equation

$$u'' = 2t^{-2}u, \quad t \geqslant 1. \tag{20}$$

Clearly, the function $a(t) = 2t^{-2}$ satisfies condition (5). However, Eq. (20) has a nontrivial L^2-solution $u(t) = t^{-1}$ which is defined for all $t \geqslant 1$ and satisfies condition (9).

This example prompts that additional hypotheses are required for proving the nonexistence of nontrivial L^2-solutions of Eq. (4). A careful inspection of the statements of Theorem 1 and Corollary 4 reveals that they both provide such hypotheses, although of a quite different nature. We note that other types of assumptions could be imposed for the same purpose (see, for instance, papers on the limit-point/limit-circle classification).

Finally, consider the third order nonlinear differential equation

$$x''' = a(t)x + f(t,x), \quad t \geqslant t_0 \geqslant 1, \tag{21}$$

where $f(t,x)$ satisfies condition (A3). Let us define the functions b, c, and d by

$$b(t) \stackrel{\text{def}}{=} \int_t^\infty a(s)\,ds, \quad c(t) \stackrel{\text{def}}{=} \int_t^\infty b(s)\,ds, \quad d(t) \stackrel{\text{def}}{=} \int_t^\infty c(s)\,ds.$$

Suppose now that condition (9) is replaced with

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} x''(t) = 0, \tag{22}$$

and assume in addition that

$$\int^\infty t^3 \left|b(t)\right|^2 dt < +\infty, \qquad \int^\infty t\left|c(t)\right|^2 dt < +\infty,$$
$$\int^\infty \left|c(t)\right| dt < +\infty, \qquad \int^\infty t^5 \left|d(t)a(t)\right|^2 dt < +\infty,$$

and

$$\int^\infty t^5 w^2(t)\,dt < +\infty.$$

Then, by repeated integration of the differential equation

$$\bigl(1 + d(t)\bigr)x''' = \bigl(1 + d(t)\bigr)\bigl(a(t)x + f(t,x)\bigr), \quad t \geqslant T_0,$$

and using condition (22), we arrive at the integral equation

$$\begin{aligned} x(t) = {} & -d(t)x(t) + 3\int_t^\infty c(s)x(s)\,ds \\ & - 3\int_t^\infty (s-t)b(s)x(s)\,ds - \frac{1}{2}\int_t^\infty (s-t)^2 d(s)a(s)x(s)\,ds \end{aligned}$$

$$- \frac{1}{2} \int_{t}^{\infty} (s-t)^2 \big(1 + d(s)\big) f\big(s, x(s)\big)\, ds.$$

It is clear now that the result analogous to Theorem 3 can be also formulated for Eq. (21), and its proof uses an argument similar to that exploited for proving Theorem 3.

Acknowledgments

The authors express their gratitude to the referee for careful reading of the manuscript and suggesting improvements in the presentation. The first author is indebted to Professor Constantin P. Niculescu of the University of Craiova for unfailing support and encouragement.

References

[1] F.V. Atkinson, Nonlinear extensions of limit-point criteria, Math. Z. 130 (1973) 297–312.
[2] M. Bartušek, J.R. Graef, On the limit-point/limit-circle problem for second order nonlinear equations, Nonlinear Stud. 9 (2002) 361–369.
[3] M. Bartušek, Z. Došlá, J.R. Graef, Nonlinear limit-point type solutions of nth order differential equations, J. Math. Anal. Appl. 209 (1997) 122–139.
[4] M. Bartušek, Z. Došlá, J.R. Graef, On the definitions of the nonlinear limit-point/limit-circle properties, Differential Equations Dynam. Systems 9 (2001) 49–61.
[5] M. Bartušek, Z. Došlá, J.R. Graef, The nonlinear limit-point/limit-circle problem for higher order equations, Equadiff 9 (Brno, 1997), Arch. Math. (Brno) 34 (1998) 13–22.
[6] J. Burlak, On the nonexistence of L^2-solutions of a class of nonlinear differential equations, Proc. Edinburgh Math. Soc. (2) 14 (1965) 257–268.
[7] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw–Hill, New York, 1955.
[8] J. Detki, On the solvability of certain second order nonlinear ordinary differential equation in $L^p(0,\infty)$, Math. Balkanica 4 (1974) 115–119 (in Russian).
[9] J. Eliaš, On the solutions of nth order nonlinear differential equation in $L^2(0,\infty)$, Math. Slovaca 32 (1982) 427–434.
[10] J.R. Graef, Limit-circle criteria and related problems for nonlinear equations, J. Differential Equations 35 (1980) 319–338.
[11] J.R. Graef, P.W. Spikes, On the nonlinear limit-point/limit-circle problem, Nonlinear Anal. 7 (1983) 851–871.
[12] J.R. Graef, P.W. Spikes, On the nonoscillation, convergence to zero, and integrability of solutions of a second order nonlinear differential equation, Math. Nachr. 130 (1987) 139–149.
[13] M.K. Grammatikopoulos, M.R. Kulenović, On the nonexistence of L^2-solutions of nth order differential equations, Proc. Edinburgh Math. Soc. (2) 24 (1981) 131–136.
[14] T.G. Hallam, On the nonexistence of L^p-solutions of certain nonlinear differential equations, Glasgow Math. J. 8 (1967) 133–138.
[15] T.G. Hallam, Asymptotic integration of second order differential equations with integrable coefficients, SIAM J. Appl. Math. 19 (1970) 430–439.
[16] P. Hartman, A. Wintner, On non-oscillatory linear differential equations, Amer. J. Math. 75 (1953) 717–730.
[17] R.M. Kauffman, T. Thomas, A. Zettl, The Deficiency Index Problem for Powers of Ordinary Differential Expressions, in: Lecture Notes in Mathematics, vol. 621, Springer-Verlag, Berlin, 1977.
[18] O.G. Mustafa, Yu.V. Rogovchenko, Global existence of solutions with prescribed asymptotic behavior for second-order nonlinear differential equations, Nonlinear Anal. 51 (2002) 339–368.
[19] O.G. Mustafa, Yu.V. Rogovchenko, Existence of square integrable solutions of perturbed nonlinear differential equations, Proc. Fourth Internat. Conf. Dynam. Systems Differential Equations, Discrete Contin. Dynam. Systems supplement volume (2003) 647–655.

[20] O.G. Mustafa, Yu.V. Rogovchenko, Limit-point type results for linear differential equations, Arch. Inequal. Appl. 1 (2003) 387–395.
[21] O.G. Mustafa, Yu.V. Rogovchenko, Limit-point criteria for superlinear differential equations, Bull. Belg. Math. Soc. Simon Stevin (2004), in press.
[22] W.T. Patula, P. Waltman, Limit point classification of second order linear differential equations, J. London Math. Soc. (2) 8 (1974) 209–216.
[23] L. Suyemoto, P. Waltman, Extension of a theorem of A. Wintner, Proc. Amer. Math. Soc. 14 (1963) 970–971.
[24] H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwocklungen willkülricher Funktionen, Math. Ann. 68 (1910) 220–269.
[25] A. Wintner, A criterion for the nonexistence of (L^2)-solutions of a non-oscillatory differential equation, J. London Math. Soc. 25 (1950) 347–351.
[26] J.S.W. Wong, Remarks on a theorem of A. Wintner, Enseign. Math. (2) 13 (1967) 103–106.
[27] J.S.W. Wong, On L^2-solutions of linear ordinary differential equations, Duke Math. J. 38 (1971) 93–97.
[28] A. Zettl, Square integrable solutions of $Ly = f(t, y)$, Proc. Amer. Math. Soc. 26 (1970) 635–639.